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A BOUND ON THE SIZE OF POINT CLUSTERS OF A
RANDOM WALK WITH STATIONARY INCREMENTS

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A bound on the size of point clusters of a random walk with stationary increments

by

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SUMMARY

Consider a random walk on $\mathbb{R}^d$ with stationary, possibly dependent increments. Let $N(V)$ count the number of visits to a bounded set V . We give bounds on the size of $N(t+V)$, uniformly in t , in terms of the behavior of N in a neighborhood of the origin.

KEY WORDS & PHRASES: *Stationary increments, point cluster, point process*

1. INTRODUCTION

Let $(\xi_n)_{n \in \mathbb{Z}}$ be a stationary sequence of random vectors in the d -dimensional Euclidean space $(\mathbb{R}^d, \mathcal{B}^d)$. The process $(S_n)_{n \in \mathbb{Z}}$, determined by

$$S_0 := 0, S_n = \xi_n + S_{n-1}, n \in \mathbb{Z},$$

is called a *random walk with stationary increments*. This definition of S_n for all $n \in \mathbb{Z}$ is uncommon but will be useful in the present context. Define the point process N by

$$N(B) := \sum_{n \in \mathbb{Z}} 1_B(S_n), \quad B \in \mathcal{B}^k.$$

We assume that the random walk is *transient*, i.e. N is finite on bounded Borel sets B .

For random walks on $\mathbb{R}^1$ with stationary, non-negative increments KAPLAN (1955) proved that $EN(t, t+h) \leq EN(-h, h)$ for real t and $h > 0$. In case the increments are independent, this inequality is a simple consequence of the Markov property (see FELLER (1970, VI.10)) and in fact $N(t, t+h)$ is stochastically dominated by $N(-h, h)$. Below we shall see that this domination does not hold without independence.

Let us now consider random walks on $\mathbb{R}^d$. Assume V is a bounded Borel set with translate $V+t := \{s+t: s \in V\}$, and suppose $V_0 := \{s-t: s, t \in V\}$ is also a Borel set. We prove that if $f \geq 0$ is a function, growing not too slowly such that

$$(1) \quad n(f(n+1)-f(n)) \geq 0 \text{ is non-decreasing}$$

then

$$(2) \quad Ef(N(V)) \leq Ef(N(V_0)).$$

The condition (1) is satisfied for e.g. $f(n) = n^\alpha$, $\alpha > 0$, or $f(n) = \log n$. If (2) were true for any non-decreasing f then $N(V)$ would be stochastically dominated by $N(V_0)$. However we prove

$$(3) \quad P(N(V) \geq p) \leq \gamma P(N(V_0) \geq p) \quad \text{where } \gamma = 2 - \frac{1}{p}$$

for $p = 1, 2, \dots$. An example will show that γ cannot be smaller without restricting V . The two results above will follow from the more general theorem 1 below. Inequality (3) could also be proved directly using the method of BERBEE (1979), theorem 2.2.3.

Suppose $0 = f(0) \leq f(1) \leq \dots$ is given. Let $c(n) := \frac{1}{n} \sum_{k=1}^n f(k)$ be a Cesaro average and let

$$h(n) := c(n) + \sup_{k \leq n} (f(k) - c(k)).$$

We shall see that (1) implies that $f - c$ is non-decreasing and then $f \equiv h$. In section 2 we show

THEOREM 1. $Ef(N(V)) \leq Eh(N(V_0))$.

This result and also (2), (3) and (5) can be improved slightly if $-V$ is a translate of V . In that case we may replace $N(V_0)$ by

$$(4) \quad \sup_{V' \ni 0} N(V')$$

where V' runs over the translates of V .

In section 3 we pay special attention to random walks on the real line. We prove for an interval $V = (t, t+h)$

$$(5) \quad P(N(V) \geq p) \leq \gamma P(N(V_0) \geq p) \quad \text{where } \gamma = \frac{3}{2} - \frac{1}{2p}$$

for $p = 1, 2, \dots$. An example shows that γ cannot be smaller.

Replacing V by $V+t$ in the inequalities does not change V_0 . As a consequence an important application of our results concerns uniform integrability. Suppose that $EN(U) < \infty$ on a neighborhood U of the origin. Using that the bounded set V is contained in a finite union of translates of U , it is proved easily from our inequalities that $N(V+t)$ is integrable, uniformly in t . This result is used in BERBEE (1979) to obtain Blackwell's theorem for stationary processes. A related integrability problem is solved

in DALEY (1971) in connection with the global renewal theorem. A condition for finiteness of $EN(U)$ can be found in LAI (1977) in terms of strong mixing. In the limit theory of semi-Markov chains very complicated integrability conditions are used (see KESTEN (1974)).

2. INEQUALITIES FOR GENERAL V

The proof of theorem 1 is based on a combinatorial lemma. Let $A := (s_0, \dots, s_n)$ be a finite sequence of points in $\mathbb{R}^k$. Define the *distant cluster* of $s \in A$ as the subsequence $A(s) \equiv A \cap (V+s)$ of points of A in $V+s$ (with the same multiplicities) and the *close cluster* as $A_0(s) \equiv A \cap (V_0+s)$. Let $n(s)$ and $n_0(s)$ denote the number of points in the distant and close cluster of s .

With f and h as in theorem 1 we have the following comparison lemma for the sizes of distant and close clusters.

LEMMA 2. $\sum_s f(n(s)) \leq \sum_s h(n_0(s))$.

Here as in the proof below the sums are over the points in A with the right multiplicities.

PROOF. Obviously for $s \in A$

$$f(n(s)) \leq c(n_0(s)) + (f(n(s)) - c(n_0(s)))^+.$$

Define

$$h_1(s, t) := \frac{1}{n(s)} c(n_0(s)), \quad t \in A(s),$$

$$h_2(s, t) := \frac{1}{n(s)} (f(n(s)) - c(n_0(s)))^+, \quad t \in A(s),$$

$$h_1(s, t) = h_2(s, t) := 0, \quad \text{otherwise.}$$

Because $n(s) = \# A(s)$ we have, rewriting sums,

$$\sum_s f(n(s)) \leq \sum_s \left(\sum_t h_1(s, t) + \sum_r h_2(r, s) \right)$$

and it suffices to prove that the term in brackets is at most $h(n_0(s))$. This term equals

$$(6) \quad c(n_0(s)) + \sum_{r:s \in A(r)} \frac{1}{n(r)} (f(n(r)) - c(n_0(r)))^+.$$

If $s \in A(r)$ then $V + r \subset V_0 + s$ so $n(r) \leq n_0(s)$. Hence (6) is at most

$$c(n_0(s)) + \sum_{r:s \in A(r)} \sup_{n \leq n_0(s)} \frac{1}{n} (f(n) - c(n_0(r)))^+.$$

The sum above is taken over $k := \# A \cap (-V+s)$ terms. If $s \in A(r)$ then $-V + s \subset V_0 + r$, so $k \leq n_0(r)$. Because c is non-decreasing $(f(n) - c(j))^+$ is non-decreasing in j . Hence (6) is at most

$$(7) \quad c(n_0(s)) + k \sup_{n \leq n_0(s)} \frac{1}{n} (f(n) - c(k))^+.$$

By the definition of c as Cesaro average, the difference

$$\frac{k}{n} (f(n) - c(k)) - \frac{k-1}{n} (f(n) - c(k-1)) = \frac{1}{n} (f(n) - f(k))$$

is non-negative for $k \leq n$ and non-positive for $k \geq n$. So the expression (7) is maximal for $k = n$. Therefore (7) and so also (6) is at most $h(n_0(s))$. $\square$

REMARK 3. If $-V$ is a translate of V we can do better than in lemma 2 by taking

$$(8) \quad n_0(s) := \sup_{V' \ni s} \# A \cap V'$$

where V' runs over all translates $V + t$ of V . Then lemma 2 holds again (note that in the proof also now $n(r) \leq n_0(s)$ and $k \leq n_0(r)$ if $s \in A(r)$). The assertion concerning (4) is obtained by following the arguments below with the obvious changes.

Theorem 1 follows from lemma 2 using the ergodic theorem as follows.

PROOF of theorem 1. Take $A := (S_0, \dots, S_n)$ and define

$$\bar{N}(B) := \sum_{k=0}^n 1_B(S_k).$$

By lemma 2

$$(9) \quad \sum_{k=0}^n f(\bar{N}(S_k+V)) \leq \sum_{k=0}^n h(\bar{N}(S_k+V_0)).$$

Choose some large constant m and define for $-\infty < k < \infty$ a stationary sequence

$$\begin{aligned} N_k^{(m)} &:= N(S_k+V) && \text{if for all } |j| \geq m \text{ holds } S_{j+k} \neq S_k+V, \\ &:= 0 && \text{else.} \end{aligned}$$

With these definitions

$$\begin{aligned} N_k^{(m)} &\leq \bar{N}(S_k+V) && \text{for } m \leq k \leq n-m, \\ &\leq 2m-1 && \text{for all } k, \end{aligned}$$

and hence

$$\sum_{k=0}^n f(N_k^{(m)}) - 2m f(2m-1) \leq \sum_{k=0}^n f(\bar{N}(S_k+V)).$$

By (9) the right hand side is dominated by

$$\sum_{k=0}^n h(\bar{N}(S_k+V_0)) \leq \sum_{k=0}^n h(N(S_k+V_0)).$$

In the last inequality we used that h is non-decreasing and $\bar{N} \leq N$. Hence

$$\sum_{k=0}^n f(N_k^{(m)}) - 2m f(2m-1) \leq \sum_{k=0}^n h(N(S_k+V_0)).$$

Divide by $n+1$, let $n \rightarrow \infty$ and apply the ergodic theorem. After taking expectations we obtain

$$Ef(N_0^{(m)}) \leq Eh(N(V_0)).$$

Let $m \rightarrow \infty$. By the monotone convergence theorem this implies the assertion. $\square$

To get (2) from (1) we apply theorem 1 and the following remark.

REMARK 4. Obviously $h \equiv f$ if and only if $f(n)-c(n)$ is non-decreasing. This property holds under (1). To see this observe that f can be expressed as $f \equiv \sum_1^\infty a_p f_p$ where $a_1 := f(1)$ and

$$(n-1)(f(n)-f(n-1)) = a_2 + \dots + a_n, \quad n \geq 2,$$

specifies the other a_p . They are non-negative by (1). Here f_p is defined by

$$\begin{aligned} f_p(n) &:= \sum_p^n \frac{1}{k-1} & n \geq p > 1 \\ &:= 1 & n \geq p = 1 \\ &:= 0 & \text{else.} \end{aligned}$$

That $f-c$ is non-decreasing is checked easily for $f \equiv f_p$. This follows then also for $f \equiv \sum_1^\infty a_p f_p$.

Inequality (3) follows from theorem 1 by using $f \equiv 1_{[p,\infty)}$ and observing that for $n \geq p$

$$(10) \quad h(n) = 1 - \frac{p-1}{n} + \frac{p-1}{p} \leq \gamma = 2 - \frac{1}{p}.$$

The constant in (3) cannot be smaller because of the following example for $d = 1$.

EXAMPLE 5. Fix some $m \geq 1$. We construct a sequence $\bar{A}$ of reals

$x_1 < y_1 < \dots < x_m < y_m < z$ and a set V such that $y_i \in x_i + V$, $z \in y_i + V$ and $(x_i + V_0) \cap \bar{A} = \{x_i\}$.

Suppose this is done. Let $A = (s_0, \dots, s_n)$ consist of $(p-1)$ -tuplets at $x_1, \dots, x_m$ and p -tuplets at $y_1, \dots, y_m, z$. Then, counting with the right multiplicities

$$\# \{s \in A: n(s) \geq p\} = m(p-1) + mp$$

$$\# \{s \in A: n_0(s) \geq p\} = mp + p.$$

If m is large the ratio γ_m of these numbers is close to $2 - \frac{1}{p}$.

To construct the probabilistic example, let $\omega := (\omega_k)_{k \in \mathbb{Z}}$ have period $n+1$ such that $\omega_i = s_i - s_{i-1}$, $1 \leq i \leq n$, and ω_0 is some very large number. Let each element of $\Omega := \{T^i \omega, 0 \leq i \leq n\}$ have equal probability. The identity ξ on Ω is stationary and the ratio of the probabilities in (3) is γ_m as above.

To construct $\bar{A}$ let $2 < p_1 < p_2 < \dots$ be primes. Take $z := 0$ and

$$y_i := p_1 \cdot \dots \cdot p_{m+i}$$

$$x_i := y_i - p_1 \cdot \dots \cdot p_i, \quad 1 \leq i \leq m,$$

and let $V := \{p_1 \cdot \dots \cdot p_i : 1 \leq i \leq 2m\}$. The only property of $\bar{A}$ that is not obvious is $(x_i + V_0) \cap \bar{A} = \{x_i\}$. Let us call products of more than m primes long and the other products short. Each $v \in V_0$ is uniquely represented as difference of two elements in V . Let v_ℓ be obtained by replacing in this difference the short products by 0. Also $(x_i)_\ell := y_i$.

Suppose $x_j \in x_i + V_0$. It is easily proved that for the long products in $x_j - x_i = v \in V_0$ we have $y_j - y_i = v_\ell$ and then we should have $v = v_\ell$. So $y_j - y_i = x_j - x_i$ and $i = j$. Similar considerations disprove y_j or $0 \in x_i + V_0$. Hence $(x_i + V_0) \cap \bar{A} = \{x_i\}$.

3. INEQUALITIES FOR INTERVALS

Let $d = 1$ and assume $V = (t, t+h)$. Let $A := (s_0, \dots, s_n)$ and take $n(s) := \# A \cap (V + s)$ as before but define $n_0(s)$ by (8). Because $-V$ is a translate of V lemma 2 holds. We get (5) from lemma 6 as in the proof of theorem 1. Counting $s \in A$ with its multiplicity, we have

LEMMA 6. $\# \{s \in A : n(s) \geq p\} \leq \left(\frac{3}{2} - \frac{1}{2p}\right) \# \{s \in A : n_0(s) \geq p\}.$

PROOF. Let $f \equiv 1_{[p, \infty)}$. Then $h(n) \leq \frac{3}{2} - \frac{1}{2p}$ for $n \leq 2p$ by (10). Hence if $n_0(s) \leq 2p$ for all $s \in A$ then the assertion follows from lemma 2.

Let $\gamma(A) := \#/\#_0$ be the ratio of the numbers at the left and right in the assertion. If $\gamma(A) \leq 1$ nothing has to be proved. Otherwise there may exist an interval $I = (x, x+h)$ with more than $2p$ points of A . We will remove

one of these points to get A' and will show $\gamma(A) \leq \gamma(A')$. Continuing this procedure we would come in finitely many steps to A'' with no such intervals I . For such A'' we already obtained the assertion and so $\gamma(A) \leq \gamma(A'') \leq \frac{3}{2} - \frac{1}{2p}$ would complete the proof.

So consider A and I as above and remove $\bar{s} \in A \cap I$ from A such that both in $(x, \bar{s}]$ and $[\bar{s}, x+h)$ at least p points of A are left. One checks easily that then $\# A' \cap V' \geq p$ if $\# A \cap V' \geq p$ for any translate V' of V . Hence in $\gamma(A) := \#/\#_0$ the removal of $\bar{s}$ causes the denominator (numerator) to decrease by (at most) 1. Because $\gamma(A) \geq 1$ we may conclude $\gamma(A') \geq \gamma(A)$. $\square$

EXAMPLE 7. The constant γ in (5) cannot be smaller than $\frac{3}{2} - \frac{1}{2p}$. To see this let $0 < \varepsilon_0 < \dots < \varepsilon_m < 1$. Let A contain p -tuplets at $5k$ and $5k + \varepsilon_k$ and $(p-1)$ -tuplets at $5k + \varepsilon_k + 1$, $0 \leq k \leq m$. With $V := (5, 6)$ the ratio γ_m of

$$\# \{s \in A: n(s) \geq p\} = (3p-1)m$$

$$\# \{s \in A: n_0(s) \geq p\} = 2p(m+1)$$

is close to $\frac{3}{2} - \frac{1}{2p}$ for large m . Here we may take $n_0(s) := \# A \cap (V_0 + s)$. Just as in example 5 we can construct a probability space where the ratio of the probabilities in (5) is γ_m .

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